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Slow viscous flow of an incompressible stratified fluid past a sphere

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The steady, uniform, horizontal flow of a vertically stratified, non-diffusive fluid over a sphere is considered. A correction to the Stokes drag formula is obtained which is valid for small values of a stratification parameter α, $Re \leqslant |\alpha|^{\frac{1}{3}}$ and $Fr^2 \leqslant |\alpha|^{-\frac{1}{3}}$. To the order of the calculations, the sphere has no tendency to rotate, nor does it experience a lift force.

1. Introduction

The flow of a vertically stratified fluid at low Reynolds number past two-dimensional objects has received previous attention from several authors. Martin & Long (1968) considered horizontal flow past a horizontal flat plate, Graebel (1969) treated a cylinder in uniform horizontal translation and Janowitz (1971) horizontal flow past a finite vertical flat plate. This present work is apparently the first attempt to deal with a closed three-dimensional body. Specifically, we consider the steady horizontal flow of non-diffusive viscous fluid past a sphere, where far upstream the velocity is uniform and density is a linear function of the vertical co-ordinate y. The governing equations and the boundary conditions for the problem are (in non-dimensional variables)

$$-\nabla p + \nabla^2 \mathbf{q} - \alpha\rho\mathbf{j} = (Re + \alpha Fr^2 \rho)\,\mathbf{q}\,.\,\nabla\mathbf{q}, \tag{1a}$$

$$\nabla\,.\,\mathbf{q} = 0, \quad \mathbf{q}\,.\,\nabla\rho = 0, \tag{1b,c}$$

$$\mathbf{q} \to \mathbf{i} \quad \text{as} \quad r \to \infty \quad (r^2 = x^2 + y^2 + z^2), \tag{2a}$$

$$\mathbf{q} = 0 \quad \text{on} \quad r = 1, \tag{2b}$$

$$\rho \to -y \quad \text{as} \quad |x| \to \infty. \tag{2c}$$

The normalization is defined as follows:

U = free-stream velocity,

a = radius of sphere,

ρ_0' = upstream density evaluated at $y = 0$,

$$\mathbf{q} = \frac{\mathbf{q}'}{U}, \quad p = \frac{ap'}{\mu U} + \frac{Re}{Fr^2}y, \quad \rho = \frac{\rho' - \rho_0'}{\rho_0' a\beta}, \quad x = \frac{x'}{a}, \quad \text{etc.},$$

$Re = \rho_0' Ua/\mu$ = Reynolds number,

$Fr = U/(ga)^{\frac{1}{2}}$ = Froude number,

$d\rho'_\infty/dy' = -\rho'_0\beta$ = constant upstream density gradient,

$\alpha = a\beta Re/Fr^2$ = non-dimensional parameter, assumed to be small,

$\mathbf{i}, \mathbf{j}$ = unit vectors in the x, y directions.

The primed quantities ρ', p' and $\mathbf{q}'$ are in physical units and have their usual hydrodynamic meanings. We seek an approximate solution to (1) and (2) subject to the conditions that $|\alpha| \ll 1$, $Re \ll |\alpha|^{\frac{1}{3}}$ and $Fr^2 \ll |\alpha|^{-\frac{1}{3}}$. It is shown in § 2 that under the latter two conditions the inertia terms can be neglected. The solutions for positive and negative upstream density gradients are formally the same, however the former may be unstable. It is further assumed that the sphere is neutrally buoyant and not constrained in any way. It is not obvious at the outset that (2*b*) is appropriate since the sphere may rotate; however, it is shown in § 5 that in fact the sphere does not spin, at least to the order to which the calculations are made.

Equation (1*c*) states that the density is constant on stream surfaces. To treat the diffusive case, the right-hand side of (1*c*) must be replaced by an inverse Péclet number times the Laplacian of ρ.

A perturbation scheme involving inner and outer expansions is used, and is similar to that used by Chang (1960). Fourier transforms are introduced in the outer problem, and the matching and the calculations of the resultant forces on the sphere proceed along the lines of Childress (1964). The main analytical results of the present investigation can be summarized as

$$\mathbf{F}' = 6\pi\mu U a\mathbf{i}(1 + B|\alpha|^{\frac{1}{3}}) + O(\alpha^{\frac{2}{3}}), \tag{3a}$$

$$\boldsymbol{\omega}' = O(\alpha^{\frac{2}{3}}), \tag{3b}$$

where $\mathbf{F}'$ is the resultant force on the sphere and $\boldsymbol{\omega}'$ its angular velocity in physical units. B is a positive constant whose value is 0·146, and is determined by the integration of a triple definite integral. Equation (3*a*) indicates that the resultant force on the sphere is a drag force to $O(\alpha^{\frac{1}{3}})$. The sphere does not rotate to $O(\alpha^{\frac{1}{3}})$; the orders of magnitude of the first terms contributing to rotation and lift have not yet been determined.

2. The perturbation method

We consider the following expansions of the dependent variables:

$$\mathbf{q} = \mathbf{h}^{(0)}(x,y,z) + |\alpha|^{\frac{1}{3}}\mathbf{h}^{(1)}(x,y,z) + |\alpha|^{\frac{2}{3}}\mathbf{h}^{(2)}(x,y,z) + \ldots, \tag{4a}$$

$$p = p^{(0)}(x,y,z) + |\alpha|^{\frac{1}{3}}p^{(1)}(x,y,z) + |\alpha|^{\frac{2}{3}}p^{(2)}(x,y,z) + \ldots, \tag{4b}$$

$$\rho = \rho^{(0)}(x,y,z) + |\alpha|^{\frac{1}{3}}\rho^{(1)}(x,y,z) + |\alpha|^{\frac{2}{3}}\rho^{(2)}(x,y,z) + \ldots. \tag{4c}$$

It is not known *a priori* that the expansions should proceed in powers of $|\alpha|^{\frac{1}{3}}$, only through matching is this proper sequence determined. Substitution of

(4) into (1) yields the following hierarchy of equations:

$$\left.\begin{aligned} -\nabla p^{(i)}+\nabla^2\mathbf{h}^{(i)} &= 0 \\ \nabla.\mathbf{h}^{(i)} &= 0 \\ \sum_{j=0}^{i}\mathbf{h}^{(j)}.\nabla\rho^{(i-j)} &= 0 \end{aligned}\right\} \text{ at } O(\alpha^{\frac{1}{3}i}), \quad i = 0, 1, \qquad (5a),(6a);\ (5b),(6b);\ (5c),(6c)$$

$$\left.\begin{aligned} -\nabla p^{(2)}+\nabla^2\mathbf{h}^{(2)} &= Re\,\mathbf{h}^{(0)}.\nabla\mathbf{h}^{(0)} \\ \nabla.\mathbf{h}^{(2)} &= 0 \\ \sum_{j=0}^{2}\mathbf{h}^{(j)}.\nabla\rho^{(2-j)} &= 0 \end{aligned}\right\} \text{ at } O(\alpha^{\frac{2}{3}}) \text{ etc.} \qquad (7a);\ (7b);\ (7c)$$

Proceeding in the usual way we attempt to solve (5)–(7) subject to the boundary conditions (2), but we find that, at $O(\alpha)^{\frac{2}{3}}$, $\mathbf{h}^{(2)}$ does not approach zero as $r \to \infty$. In other words, 'Whitehead's Paradox' (cf. Van Dyke 1964, p. 153) occurs; so we can conclude that the expansions (4) do not provide a uniformly valid approximation as $r \to \infty$. The expansions (4), which are valid in the neighbourhood of the sphere, are referred to as the inner expansions. Outer expansions uniformly valid in the neighbourhood of $r = \infty$ are now introduced:

$$\mathbf{q} = \mathbf{i}+|\alpha|^{\frac{1}{3}}\mathbf{g}^{(1)}(\tilde{x},\tilde{y},\tilde{z})+|\alpha|^{\frac{2}{3}}\mathbf{g}^{(2)}(\tilde{x},\tilde{y},\tilde{z})+\ldots, \qquad (8a)$$

$$p = |\alpha|^{\frac{1}{3}}p^{(0)}+|\alpha|^{\frac{2}{3}}p^{(1)}(\tilde{x},\tilde{y},\tilde{z})+|\alpha|\,p^{(2)}(\tilde{x},\tilde{y},\tilde{z})+\ldots, \qquad (8b)$$

$$\rho = -|\alpha|^{-\frac{1}{3}}\tilde{y}+\rho^{(1)}(\tilde{x},\tilde{y},\tilde{z})+|\alpha|^{\frac{1}{3}}\rho^{(2)}(\tilde{x},\tilde{y},\tilde{z})+\ldots, \qquad (8c)$$

where the outer variables are defined by

$$\tilde{x} = |\alpha|^{\frac{1}{3}}x, \quad \tilde{y} = |\alpha|^{\frac{1}{3}}y, \quad \tilde{z} = |\alpha|^{\frac{1}{3}}z. \qquad (9)$$

The stretching factor given by (9) is suggested by the magnitudes of the neglected body-force term and viscous term in (5), which for large r are in the ratio $\alpha : r^{-3}$ when y is of order unity. When $r = O(\alpha^{-\frac{1}{3}})$ it can be expected that the inner expansions (4) cease to be accurate.

Substitution of expansions (8) into (1) yields the equations for the outer problem:

$$-\tilde{\nabla}p^{(1)}+\tilde{\nabla}^2\mathbf{g}^{(1)}\mp\rho^{(1)}\mathbf{j} = |\alpha|^{-\frac{1}{3}}[Re-|\alpha|^{\frac{2}{3}}\tilde{y}\,Fr^2]\,\partial\mathbf{g}^{(1)}/\partial\tilde{x}, \qquad (10a)†$$

$$\tilde{\nabla}.\mathbf{g}^{(1)} = 0, \quad \partial\rho^{(1)}/\partial\tilde{x} = \mathbf{g}^{(1)}.\mathbf{j}, \qquad (10b, c)$$

$p^{(0)}$ being simply the hydrostatic contribution

$$p^{(0)} = \tfrac{1}{2}\tilde{y}^2. \qquad (11)$$

The inertia terms in (10*a*) can be neglected provided that $Re \ll |\alpha|^{\frac{1}{3}}$ and $Fr^2 \ll |\alpha|^{-\frac{1}{3}}$ as stated in § 1. Under the transformation (9) the sphere shrinks to a point as $\alpha \to 0$, so (2*b*) must be relaxed when solving (10). Similarly we relax (2*a*) in the inner problem, and instead require that the inner and outer expansions agree term by term in some overlap region where both expansions are valid, namely $r = O(|\alpha|^{-\frac{1}{3}\sigma})$, where $0 < \sigma < 1$.

† The upper and lower signs used here and throughout refer to the cases of negative and positive upstream density gradient, respectively.

3. The solutions for $\mathbf{h}^{(0)}$, $p^{(0)}$ and $\rho^{(0)}$

We seek a solution to (5) for which the no-slip condition (2*b*) is still enforced and the boundary conditions at infinity are the first terms of the outer expansions (4), i.e.

$$\mathbf{h}^{(0)} \to \mathbf{i}, \quad p^{(0)} \to 0, \quad \rho^{(0)} \to -y \quad \text{as} \quad r \to \infty. \tag{12a,b,c}$$

Because $\rho^{(0)}$ is uncoupled from the velocity and pressure, $\mathbf{h}^{(0)}$ and $p^{(0)}$ are just the solution to the standard Stokes problem without stratification, and by symmetry it can be argued that the sphere does not spin to this order, so (2*b*) is appropriate.

The solution is

$$\mathbf{h}^{(0)} = \mathbf{i} - \frac{3}{2}\left(\frac{\mathbf{i}}{r} - \nabla \frac{x}{2r}\right) + 1/4 \nabla \frac{\partial}{\partial x} \frac{1}{r},$$

$$p^{(0)} = -3x/2r^3,$$

$$\rho^{(0)} = \left(\frac{1}{r} - 1\right)\left(1 + \frac{1}{2r}\right)^{\frac{1}{2}} y.$$

We define intermediate variables $r_\sigma = |\alpha|^{\frac{1}{3}\sigma} r$ and $x_\sigma = |\alpha|^{\frac{1}{3}\sigma} x$, where $0 < \sigma < 1$, and rewrite $\mathbf{h}^{(0)}$ in these variables:

$$\mathbf{h}^{(0)}(x_\sigma, r_\sigma; |\alpha|) = \mathbf{i} - \frac{3}{2}\left(\frac{\mathbf{i}}{r_\sigma} - \nabla_\sigma \frac{x_\sigma}{2r_\sigma}\right) |\alpha|^{\frac{1}{3}\sigma} + O(|\alpha|^\sigma), \tag{13}$$

for later reference.

4. The solutions for $\mathbf{g}^{(1)}$, $p^{(1)}$ and $\rho^{(1)}$

Equations (10) are now solved with the right-hand side of (10*a*) replaced by $6\pi\delta(\tilde{\mathbf{r}})\,\mathbf{i}$, which is the singularity corresponding to the Stokes drag. The use of this singularity ensures that the inner and outer expansions will match in the overlap domain (cf. Childress 1964). Introducing three-dimensional Fourier transforms $\mathbf{\Gamma}(\mathbf{k})$, $\Pi(\mathbf{k})$ and $R(\mathbf{k})$, defined by

$$\mathbf{g}^{(1)} = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} e^{i\mathbf{k}.\tilde{\mathbf{r}}}\mathbf{\Gamma}(\mathbf{k})\, d\mathbf{k}, \tag{14a}$$

$$p^{(1)} = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} e^{i\mathbf{k}.\tilde{\mathbf{r}}}\Pi(\mathbf{k})\, d\mathbf{k}, \tag{14b}$$

$$\rho^{(1)} = \frac{1}{8\pi^3}\int_{-\infty}^{\infty} e^{i\mathbf{k}.\tilde{\mathbf{r}}}R(\mathbf{k})\, d\mathbf{k}, \tag{14c}$$

and substitution into (10) leads formally to the system

$$\pm R\mathbf{j} + i\mathbf{k}\Pi + k^2\mathbf{\Gamma} = -6\pi\mathbf{i}, \tag{15a}$$

$$\mathbf{k}.\mathbf{\Gamma} = 0, \quad ik_1 R = \mathbf{\Gamma}.\mathbf{j}, \tag{15b,c}$$

where

$$\mathbf{k} = k_1\mathbf{i} + k_2\mathbf{j} + k_3\mathbf{k}'.$$

Solving for the transformed dependent variables, we find

$$\mathbf{\Gamma}(\mathbf{k}) = -\frac{6\pi}{k^2}\left\{\mathbf{i} + \frac{k_1k_2\mathbf{j} - k_1(1 \pm ik_1k^2)\,\mathbf{k}}{k^2(1 \pm ik_1k^2) - k_2^2}\right\}, \tag{16a}$$

$$\Pi(\mathbf{k}) = \frac{6\pi ik_1(1 \pm ik_1k^2)}{k^2(1 \pm ik_1k^2) - k_2^2}, \tag{16b}$$

$$R(\mathbf{k}) = \frac{\pm 6\pi k_1k_2}{k^2(1 \pm ik_1k^2) - k_2^2}. \tag{16c}$$

Substitution of (16) into (14) yields the formal solution. In the next section we extract the part of $\mathbf{g}^{(1)}$ which contributes to the drag on the sphere.

5. Matching and the calculation of drag

To calculate the resultant force on the sphere and its angular velocity, it is sufficient to expand the inversion of (16a) about $\tilde{\mathbf{r}} = 0$. It is convenient to consider the integral

$$\frac{1}{8\pi^3}\int_{-\infty}^{\infty}(\mathbf{\Gamma} - \mathbf{\Gamma}_s)\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}\,d\mathbf{k} \quad \text{as} \quad \tilde{\mathbf{r}} \to 0, \tag{17}$$

where $\mathbf{\Gamma}_s$ is the Fourier transform of the fundamental solution for the velocity field of the Stokes equations, and is given by

$$\mathbf{\Gamma}_s(\mathbf{k}) = -\frac{6\pi}{k^2}\left\{\mathbf{i} - \frac{k_1\mathbf{k}}{k^2}\right\}. \tag{18}$$

To evaluate (17) we divide the region into two parts: $0 \leqslant k \leqslant \tilde{r}^{-\epsilon}$ and $k > \tilde{r}^{-\epsilon}$, where $0 < \epsilon < 1$. Then as $\tilde{r} \to 0$,

$$\frac{1}{8\pi^3}\int_{-\infty}^{\infty}(\mathbf{\Gamma} - \mathbf{\Gamma}_s)\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}\,d\mathbf{k} = \frac{1}{8\pi^3}\int_{k \leqslant \tilde{r}^{-\epsilon}}(\mathbf{\Gamma} - \mathbf{\Gamma}_s)\,d\mathbf{k} - \frac{3i}{4\pi^2}\int_{k > \tilde{r}^{-\epsilon}}\frac{(k_2^2\mathbf{k} - k_2k^2\mathbf{j})}{k^8}\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}\,d\mathbf{k} + \ldots, \tag{19}$$

where the dots denote terms of smaller order in $\tilde{r}$. The first term on the right side of (19) is some constant vector, say $\mathbf{B}$, and the second term on the right side of (19) we shall call $\mathbf{v}^*$. Now by definition we can write (19) as

$$\mathbf{g}^{(1)} - \mathbf{A} = \mathbf{B} + \mathbf{v}^* \quad \text{as} \quad \tilde{r} \to 0, \tag{20}$$

where

$$\mathbf{A} = -\frac{3}{2}\left(\frac{\mathbf{i}}{\tilde{r}} - \tilde{\nabla}\,\frac{\tilde{x}}{2\tilde{r}}\right).$$

Rewriting (20) in intermediate variables and using (8a) we have

$$\mathbf{q}(\mathbf{r}_\sigma; |\alpha|) = \mathbf{i} - \frac{3}{2}\left(\frac{\mathbf{i}}{r_\sigma} - \nabla_\sigma\frac{x_\sigma}{2r_\sigma}\right)|\alpha|^{\frac{1}{3}\sigma} + |\alpha|^{\frac{1}{3}}(\mathbf{B} + \mathbf{v}^*) + o(|\alpha|^{\frac{1}{3}}). \tag{21}$$

Comparing (21) with (13) it is seen that the inner and outer expansions are matched to $O(|\alpha|^{\frac{1}{3}\sigma})$. The leading term in the outer expansion of $\mathbf{q}$ which is not yet matched is $O(|\alpha|^{\frac{1}{3}})$. From (21) it follows that the inner solution $\mathbf{h}^{(1)}$ which satisfies (6) should also satisfy the boundary conditions

$$\mathbf{h}^{(1)} = 0 \quad \text{on} \quad r = 1, \tag{22a}$$

$$\mathbf{h}^{(1)} \to \mathbf{B} + \mathbf{v}^* \quad \text{as} \quad r \to \infty. \tag{22b}$$

Again it is not known *a priori* that the right side of $(22a)$ should not be replaced by $\boldsymbol{\omega} \times \mathbf{r}$, but it will be shown shortly that $(22a)$ is appropriate.

It is shown in the appendix that $\mathbf{B}$ and $\mathbf{v}^*$ have the following properties:

$$\mathbf{v}^*(0,0,0) = \nabla^2\mathbf{v}^*(0,0,0) = \nabla \times \mathbf{v}^*(0,0,0) = 0, \tag{23a, b, c}$$

$$\mathbf{B} = B\mathbf{i} = 0{\cdot}146\mathbf{i}. \tag{23d}$$

The net force acting on the sphere and its angular velocity, given by (3), follow directly from Faxen's laws (Happel & Brenner 1965, p. 67), which state that (in physical units)

$$\mathbf{F}' = 6\pi\mu a[\mathbf{q}'_\infty]_0 + \mu\pi a^3[\nabla'^2\mathbf{q}'_\infty]_0, \tag{24a}$$

$$\mathbf{T}' = 8\pi\mu a^3\{\tfrac{1}{2}[\nabla' \times \mathbf{q}'_\infty]_0 - \boldsymbol{\omega}'\}, \tag{24b}$$

where T' is the torque and the subscript zero implies evaluation of the function at the origin as if the sphere were not present. Substitution of $(12a)$, $(8a)$, $(22b)$ and $(23a, b, d)$ into $(24a)$ gives $(3a)$ directly. In the approximation that the fluid has negligible inertia, we require the torque T' to be zero, which leads to $(3b)$ after inserting $(23c)$ into $(24b)$.

6. Extension of results to axially symmetric bodies

For an axially symmetric body with Stokes drag $D'_0\mathbf{i}$, the same analysis applies, but we replace the singularity $-6\pi\delta(\tilde{\mathbf{r}})\,\mathbf{i}$ by $-D_0\delta(\tilde{\mathbf{r}})\,\mathbf{i}$ when solving the outer problem (cf. Chang 1960). In the drag formula $(3a)$ the factor $6\pi\mu Ua$ is replaced by D'_0 and B is replaced by $BD_0/6\pi$; $(3b)$ remains unchanged.

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Appendix

To evaluate $\mathbf{B}$, the first term on the right side of (19), we notice that

$$\mathbf{B} = \frac{1}{8\pi^3}\int_{-\infty}^{\infty}(\mathbf{\Gamma} - \mathbf{\Gamma}_s)\,d\mathbf{k} = \frac{3}{4\pi^2}\int_{-\infty}^{\infty}\frac{k_1k_2^2\mathbf{k} - k_1k_2k^2\mathbf{j}}{k^4[k^2(1 \pm ik_1k^2) - k_2^2]}\,d\mathbf{k}. \tag{A 1}$$

The imaginary part of the integrand does not contribute to the integral since it is odd in k_1. In the remaining, real part, the $\mathbf{j}$ and $\mathbf{k}'$ components are also odd. For the $\mathbf{i}$ component we transform to spherical co-ordinates and integrate once with respect to k, obtaining the integral

$$\mathbf{B} = \mathbf{i}\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin^{\frac{14}{3}}\theta\cos^{\frac{5}{3}}\phi\sin^2\phi}{(1 - \sin^2\theta\sin^2\phi)^{\frac{2}{3}}}\,d\theta\,d\phi. \tag{A 2}$$

Referring to Gradshteyn & Ryzhik (1965), (A 2) can be expressed in terms of Γ functions:†

$$\mathbf{B} = \mathbf{i}\frac{1}{4}\frac{\Gamma(\frac{4}{3})\,\Gamma(\frac{7}{6})}{\Gamma(\frac{8}{3})\,\Gamma(\frac{11}{6})} = 0{\cdot}146\mathbf{i}.$$

To obtain (23*a*), (23*b*) and (23*c*), we use

$$\mathbf{v}^* = -\frac{3i}{4\pi^2}\int_{k>\tilde{r}^{-\epsilon}} \frac{(k_2^2\mathbf{k} - k_2 k^2\mathbf{j})\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}\,d\mathbf{k}}{k^8}, \tag{A 3}$$

$$\nabla^2\mathbf{v}^* = +\frac{3i}{4\pi^2}\int_{k>\tilde{r}^{-\epsilon}} \frac{(k_2^2\mathbf{k} - k_2 k^2\mathbf{j})\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}\,d\mathbf{k}}{k^6}, \tag{A 4}$$

$$\nabla\times\mathbf{v}^* = -\frac{3}{4\pi^2}\int_{k>\tilde{r}^{-\epsilon}} \frac{(k^2\mathbf{k} - k_2)\,k^2\mathbf{j}}{k^8}\times\mathbf{k}\,e^{i\mathbf{k}.\tilde{\mathbf{r}}}\,d\mathbf{k}. \tag{A 5}$$

Equations (23*a*), (23*b*) and (23*c*) follow trivially by setting $\tilde{\mathbf{r}} = 0$ in the integrands of (A 3), (A 4) and (A 5) and noting that the integrands are all odd.

REFERENCES

CHANG, I.-D. 1960 Stokes flow of a conducting fluid past an axially symmetric body in the presence of a uniform magnetic field. *J. Fluid Mech.* **9**, 473.

CHILDRESS, S. 1964 The slow motion of a sphere in a rotating viscous fluid. *J. Fluid Mech.* **20**, 305.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1965 *Tables of Integrals, Series, and Products*, pp. 389, 849. Academic.

GRAEBEL, W. P. 1969 On the slow motion of bodies in stratified and rotating liquids. *Quart. J. Mech. Appl. Math.* **22**, 39.

HAPPEL, J. & BRENNER, H. 1965 *Low Reynolds Number Hydrodynamics*. Prentice-Hall.

JANOWITZ, G. S. 1971 The slow transverse motion of a flat plate through a non-diffusive stratified fluid. *J. Fluid Mech.* **47**, 171.

MARTIN, S. & LONG, R. R. 1968 The slow motion of a flat plate in a stratified fluid. *J. Fluid Mech.* **31**, 669.

VAN DYKE, M. 1964 *Perturbation Methods in Fluid Mechanics*, p. 153. Academic.

† This step was suggested by a referee.